SAMPLE PROBLEM BLOCK STRATEGIES

- Page T1, linked blocks 1, 2, Section TI
  - Overview: The basic "content" meant to be "covered" by these two blocks, and the accompanying explanations/discussions (if required) is
    * the idea of an "angle" representing a *numerical* characterization of the fundamentally *geometical* notion of direction in a plane;
    * the motivation for introducing such an alternate numerical characterization in the first place;
    * the similarities to the analogous situation of parametrizing points on a line (geometric objects) using Cartesian coordinates (the numerical characterization)
    * the non-analogous (multiple-value) aspect of angular parametrization

    MANY students in the sessions take multiple iterations to get through this page, which is the first in the worksheet materials. There are a significant number of more detailed (and more general) points to be made in the explanations given to such students. Students who have completed this worksheet also often stop to listen to many of these slightly more general discussion points, despite having already successfully answered all the questions on the page.

    More details on these aspects of this worksheet, together with expanded versions of the points listed above, are given in what follows.
  - Block 1: The idea of an "angle" as a numerical, alternate way of characterizing the fundamentally geometric idea of direction in the plane:
    * Motivation: the idea of communicating information "at a distance" allowing geometrical information (direction in the plane) to be specified without having to be "be in the same room" as the person using a finger/arrow/etc

to characterize that direction “geometrically”. This also serves to introduce/discuss, by means of an illustrative example, the idea that students should try to get in the habit of asking why/ what the point is when a new concept/perspective is introduced to them, and start building up a recognition that having multiple possible ways of looking at a given problem is a good, not a bad, thing (typically, for students for whom math consists essentially entirely of a vast collection of problem solution templates to be memorized, any such new information dealing with an area “already covered” by old information represents an unpleasant onerous new memorization task, and hence is automatically viewed as negative).

* In discussing the underlying conceptual ideas (necessary for typically 3/4 or more of the students in the sessions) there is a natural opening to include in those discussions a review of the related idea of characterizing points on a line (basic geometric objects) by numbers (Cartesian coordinates). This is, in fact, something many students have a somewhat shaky understanding of the point of. The example of coordinates of points on a line allows the session leader to highlight for students, in a context they typically find “more familar” than that of angular measure, the necessity of establishing conventions (in the case of assigning Cartesian coordinates: what point is assigned the number zero, which direction is chosen to be the positive and which the negative one, and what units are going to be used before) and emphasizing that these conventions have to be established before the “geometric$< -- >$numerical” translation scheme can actually be used. [This analogue discussion has, as well, the added side benefit of helping to improve many students’ less-than-ideal understanding of the idea behind, and use of, Cartesian coordinates, in our experience.]

* For students who have trouble to begin with, the resulting conceptual discussion also provides an opportunity to emphasize how to, in general, go about approaching/assessing new information and perspectives when they arise in mathematics, and how the relation between new and old perspectives can be used to solve problems through understanding rather than the memorization of a solution template covering each problem type. Here, specifically:

helping students recognize that an understanding of how it was the number (angle) characterizing the direction in question would have been obtained in the first place, starting from the fundamental geometrical description of that direction, allows them to naturally understand how to recast the numerical value initially given into an alternate form ($\pm 2\pi$ times some fraction) where its basic geometric meaning is more immediately evident. Because students have typically been taught to use templates, rewriting something compact in a less compact form is something that almost never occurs to them, even in a situation where the less compact form makes the problem much simpler intuitively. This is a very useful general lesson, which doesn't have to be described as such in an abstract way.

- Block 2: The feature (not shared with the coordinates-of-points-on-a-line analogue) that the angle characterizing a geometrical direction is not unique (and why).
  * Discussion of this section is VERY commonly required, even after the students have successfully navigated Block 1.
  * To orient their thinking in terms of how generally to approach new material/problem types, it is useful to do what has not typically been done for them and explain what the general issue is (after conventions are set, two different Cartesian coordinates definitely correspond to two different (geometrical) points, whereas, with angles, two angles can be different but still correspond to the same geometrical direction).
  * The problems in this block ask the students to figure out, given two different angles, whether these represent the same direction or different directions. Students in the sessions are typically quite unused to framing problems they meet in such a slightly more general manner, and the problems in this block are meant to provide an opportunity to discuss the idea that moving away from a "what's the template for solving a problem like this?" approach to a more flexible, conceptually based one can make life easier for the student, producing simpler solutions and ones easily arrived at without the need to call up some memorized template out of memory (and hence without having had to have previously met "this type of problem" and memorized the relevant solution template).
  * A bit more explicitly: students with a mathematics = memorized solution templates education, even after suc-

cessfully finishing Block 1 using geometrical understanding, will quite often default to attempting to convert this new solution skill into a template for use in approaching the new problems in this second block (by identifying the direction corresponding to each of the two angles in the problem and seeing if they can tell whether or not these are the same). This is, of course, not incorrect, but not as efficient as keeping the underlying idea of how it is that two numerically different angles could end up representing the same direction in mind (the larger corresponds to adding some number of complete counter-clockwise revolutions to the smaller).

* This alternate perspective is one it is often useful to save until after students have employed the alternate approach based on the solution method they've learned for dealing with the questions in Block 1. It's also psychologically useful to be able to discuss the alternate (more efficient) approach in a context where the student may have answered the Block 2 questions correctly on their own, albeit using the less efficient approach (if this is the case, it can usually be seen from the rough work on the worksheet page).

* This Block thus represents a natural point at which to emphasize something students with template-based mathematics educations are often not familiar with, namely that there is typically not just one way to solve a given problem, and that any solution which is logically correct is an entirely acceptable one. *Many of the students in the sessions have, in fact, had the experience of having had marks actually deducted in pre-university classes when they provided a logically correct solution to a problem using a method which was not the one the teacher taught them to use for solving that type of problem.* It is very common to find students saying things like "I'm pretty sure I got the right answer, but I'm not sure if method I used to get it is the right one." Such students need help to begin to feel naturally "licensed" to pursue solutions to problems based on their own understanding and assessment of the problem, rather than treating mathematics as a large-scale exercise in template memorization.

* Often, in the course of the discussions for this block, opportunities naturally arise to comment briefly on some slightly more general points, e.g., the utility of approaching new problems flexibly, the idea that having multiple

possible approaches is an advantage and not an unpleasant additional burden on memory, and especially the idea that, although one alternative might turn out to provide a much easier method for solving a particular problem, that doesn't mean the other less efficient methods are wrong. The fact that the alternative that focusses most directly on the underlying idea (same direction if they differ by a complete number of revolutions) in this case turns out to also be far and away the fastest one to use (there is no need to actually figure out what each individual angle corresponds to geometrically if all one wants to do is know whether they correspond to the same or different directions) also serves as a useful example for starting to implicitly "train" them away from automatic template use.

- Embedded precursor(s): (primarily Block 1, but sometimes arising in Block 2 as well, depending on how the students approach the Block 2 questions): fractions.
  * Students attending the sessions typically have an at-best-poor (and often essentially non-existent) intuitive understanding of division, fractions, and hence of manipulations involving division and/or fractions.
  * Here, e.g., Question 1 involves an angle (numerical characterization) whose underlying geometrical characterization is: counter-clockwise rotation starting from the zero direction by 5/16 of a full revolution. Students not infrequently, even after finding this equivalent geometric characterization, may have trouble determining which quadrant this direction corresponds to because they're not sure whether 5/16 lies between 0 and 1/4, 1/4 and 1/2, 1/2 and 3/4 or 3/4 and 1. This type of problem is even more common in the case of Questions 4 through 6, where the geometrical characterizations of the directions in question involve the fractions 13/6, -14/10 and 9/10 of a revolution and a successful comparison to 0, 1/4, 1/2, 3/4 etc. is easiest if both the fraction of a revolution of the angle in question and the analogous fraction characterizing the direction in which the coordinate axes (which divide the plane into its four quadrants) point are changed into a form where all the fractions have a common denominator.
  * If a student has such a problem with comparing fractions (which basically ensures they will have even more

significant problems with addition and subtraction involving fractions, not to mention other more complicated operations, involving fractions) this will come up automatically here and can be dealt with briefly at this point. This helps when the student comes back to arithmetic, algebra and division/fraction properties on pages 1 and 2 of the Algebra section (which is not done until at least the first half of the trig section has been covered).

  * A lot of students DO get stuck over this problem with fractions, and this provides a natural opportunity to briefly explain the intuitive meaning of fractions, the possibility of interpreting n/m as n copies of 1/m, and thus explaining to the large number of students who have never had this explained to them before the reason for converting fractions to forms involving common denominators, and why it is that this is an intuitively sensible thing to do.
  * The natural things to discuss here are the basic idea of fractions and division, and the intuitive idea behind recasting fractions in common-denominator form when adding, subtracting or comparing two or more fractions. As with all the "precursor" elements in the various blocks, the discussions that arise from the precursor elements are meant to be, though conceptual, relatively limited in scope, serving as starters for the broader discussions that will occur in connections with later worksheets. Thus, e.g., here one would typically NOT discuss any other aspects of arithmetic involving fractions (unless one of the students taking part in the discussion happened to bring up such a question or questions naturally).

- Page A4, linked blocks 1, 2, Section A IV
  - Overview: The basic content meant to be covered, and goals meant to be accomplished, in these two blocks and their accompanying explanations/discussions (if required), are:
    * To make sure students understand the general idea of what it means to say that a given number is a solution to a given equation.
    * To make sure students don't suffer from (and, if they do, are broken of) the common reflex of assuming that to

solve any problems involving the word "solution", they are going to have to immediately launch into some sort of algebraic manipulation.

* To make sure students understand that equations in general can have no solutions, any finite number of solutions, or even an infinite number of solutions.
* To make sure students can recognize composite functions as composite, and identify the simpler functions which comprise the individual steps out of which the more complicated composite function is constructed, and how such a recognition helps in approaching the problem of finding solutions involving such composite functions.
* To provide a further opportunity to emphasize the distinction between equations (a general algebraic concept) and identities (a specific sub-class of equations, typically arising in different situations than do non-identity equations). As discussed in more detail below, the primary discussion of this issue (and important related issues, also discussed in more detail below) will have typically arisen already earlier in the session. The point is thus meant only to be mentioned again briefly here, by way of emphasis. It *is*, however, meant to be explicitly mentioned, since the distinction is important, and one we find most students attending our sessions have never had made clear to them prior to the discussion earlier in the session.

Somewhat more detail on the structure and goals of these two blocks are given in what follows.

- Problems 1-3, Block 1: These are meant to provide a context in which the more general idea that a given equation might have no solutions or multiple solutions comes up naturally, in what will appear to most students a relatively familiar context (even though many of the students encounter some problems successfully anwsering these questions the first time through).
- Problems 4-6, Block 1: These are meant to make sure that students understand that being asked to determine if a given number is a solution of a given equation is one of the easiest problems they could possibly be given, since it requires nothing other than substitution and evaluation. Remarkably, the majority of the students in the sessions have

some trouble with questions 5 and 6 because, asked whether what the number they've been given is a solution of the equation in question, they immediately attempt to algebraically manipulate the equation in question to find all of its solutions and, of course (since the questions are designed to make this impossible to accomplish, precisely to deal with this issue), are unable to do so. The ensuing discussion tends to be a real eye-opener for such students.

- Problem 7, Block 1: This is designed so that it *can* be solved algebraically by students, using techniques they will usually know from high-school. They do, though, need to first have the insight that choosing the new variable $y = x^2$ will convert the equation to $y^2+2y+5 = 0$, whose solutions can be found using the quadratic formula and then recognize that, since the discriminant of the new quadratic in y is negative, no real number solutions exist for y, and hence no real solutions exist for x either. The problem is, however, designed so that the solution is easier, and more immediately obvious, if the student approaches it flexibly at the start, assessing the nature of the less familiar fourth order equation, and not just immediately launching into algebraic manipulation. The fact that the LHS consists of a sum of two zero or positive quantities and one positive quantity, and hence that the LHS is ≥ 5, is obvious to all students, once one asks them to explore a bit by thinking about the contributions from the individual terms to the LHS, and this insight, coming from them, rather than the session leader (even if in response to a somewhat leading question from the session leader) seems to have quite a strong eye-opening effect on students who've at all got stuck on this question. This is all in the service of the larger goal of trying to implicitly train the students out of automatic immediate algebraic manipulation any time they meet a solution problem.
- Problems 8-10 (the first three problems of Block 2): These are meant to further cement the more general understanding that one can determine whether a given number is the solution of a given equation without having to produce, by algebraic manipulation, the full set of solutions to the equation. This is done by ensuring that the problems involve some element or elements not known to the students (in Problem 8 the values of the coefficients a and b, in Problems 9 and 10 the specific algebraic form of the function f). The "limitation" in the information provided is deliberately

"milder" in Problem 8 than in Problems 9 and 10.

- Problems 11-13, Block 2: These are meant to provide more work on solutions, now in the context of slightly more complicated, but fully known, functions. The functions are, by design, composite, because our experience is that the high-school functions courses most incoming students have taken do a very poor job of providing students with an ability to assess how a more complicated function is assembled from its more elementary parts. A very poor understanding of the nature of composition/multi-step functions seems to be particularly common. These questions are designed to be such that the full set of solutions, in all three cases, is easily determinable from what the students know already, having previoulsy worked through the first five Trig worksheets, provided they recognize the two-step composite nature of the functions in question and use what they know about each of the simpler individual steps to work backwards from the given second-step output to the second-step inputs required to get this output, to the first-step inputs (if these exist) which would lead to any of the first-step outputs, which would then become, via the composition process, these second-step inputs. The problems are also meant to re-emphasize the lessons learned from the discussions connected to the earlier questions in Section IV. Explicitly, the questions are designed so that

    * for the first two questions (Problems 11 and 12), the exact numerical values for all solutions could be easily obtained by the students based on what they will already know about the trig functions from the trig worksheets they will have already completed;

    * for the third question (Problem 13), they would be able to easily see that an infinite number of solutions exist, even though they would not be able to give their precise numerical values, since (by design) angles θ such that $tan(\theta) = -2$ are not among the "special angles", studied earlier in the sessions, for which exact values of the trig functions can be determined using elementary geometry.

  A very high percentage of students get stuck on these questions, frequently because they have not been given the skills required to effectively analyze, and understand the structure of, even slightly more complicated functions like these.

- It is frequently useful, therefore, and usually hugely illuminating to the students, to have, at this point, a fairly extended discussion of the way in which complicated functions are constructed, starting from a small number of elementary/basic functions via the "complicating steps" of addition, subtraction, multiplication, division, and composition (the construction of the inverse of an already familiar function can also be included as a further final example of a "complicating step"). This discussion is especially crucial given the shortcomings in the implementation of the high-school functions course that are evident from the poor understanding of functions we see in many of our incoming students, and the importance of the understanding of how complicated functions are constructed from their simpler parts in allowing students to handle the material on differentiation in their first-year calculus courses.

- Problem 14 (the last problem of Block 2): This is meant to further emphasize the importance of being willing to explore a bit a problem whose solution is not immediately obvious. There are actually quite a number of shorter blocks scattered at various places earlier in the worksheets designed with this same aim in mind. Students who've been trained in a template-based manner typically respond to a question they don't recall learning a template to handle by implicitly thinking things like "I haven't seen a problem like this before" or "I don't remember how you're supposed to solve a problem like this" and freezing up when no template comes to them. They typically have not been confronted with problems which are unfamiliar in form but involve only functions they actually know and understand, and have no practice with approaching such problems by exploring them and trying out typical strategies for doing so experienced (or better-trained) students/people would naturally consider. The earlier blocks are designed to get them to change their response to meeting an "unfamiliar" problem from one of freezing, to one of asking "is there any function/operation in the probem I've never met before" and, if the answer is no (as it always will be in the context of a mathematics class), then recognizing that this means they have to explore a bit. The different blocks are structured so one or two of the most common exploration strategies will come up naturally in the discussions connected to each such "exploration block", with different blocks designed to build up a larger repetoire of such commonly used exploration strategies/rules of thumb. Problem 14 in this block is designed to emphasize this same point. In general, session leaders are essentially NEVER supposed to solve ANY of the worksheet problems for the students, but there are a small number of "secret" exceptions, where a solution can be shown after the student has tried an iteration or two. This is one of those. Students almost always try to solve this problem by engaging (unsuccessfully) in algebraic manipulation. Often (but not always) it is then enough to suggest to them that, having found no obvious way to proceed, this means that they ought to "explore", e.g., by suggesting they remind themselves of what they know about the functions, $y = x$ and $y = cos(x)$, occurring on the two sides of the equation, maybe drawing a graph for each. Sometimes, one has to go a bit farther and get them to instruct the session leader on how to sketch these on the board. At this point, almost every student's jaw drops at how obvious this minor exploration makes the problem become, and this seems to provide a powerful lesson on the importance of responding to "that's not familiar" with "ah, that means I'm going to have to try a bit of exploration on this problem".

- Regarding the issue of distinguising between the general concept of an equation and the more restricted one of an identity: this is meant to only be brought up in passing, as part of the general framing of the discussion of the idea (and accompanying illustrations) that an equation might have no solution, or any number of solutions. It then serves both to help in framing this point and as review/reminder, since there is a block on the 7th page of the Trig section, involving trig identities, designed to trigger an earlier, more extensive discussion of this point. Typically students will have already worked through this earlier trig identity block and taken part in the associated discussion(s) (such discussion(s) turns out to be necessary for almost all students attending the sessions). Part of that earlier discussion involves focussing on an important aspect of identities we've found remarkably few students have previously had clearly explained to them. Explicitly, they have rarely had pointed out to them that there are two very common, very different types of identities:
  (i) Those whose two sides provide alternative representations of the same function, one being more useful for some problems, the other more useful for other problems. A simple example of this issue is the identity $(x^2 + 1)^2 =$

$x^4 + 2x^2 + 1$, which provides two representations of the same function $f(x)$, the LHS version being more convenient were one to meet $f(x)$ in the more complicated context $g(x) = \sqrt{f(x)}$, the RHS version being more convenient were one to meet it in the more complicated context $h(x) = f(x) - x^4 - 2x^2$.

(ii) "New-from-old" identities, where the "new" and "old" sides have different roles and one needs to know which side is meant to be thought of as the "new" side and which the "old" side (containing what is to be thought of as already known information to be used in more easily getting the output information represented by the "new" side).

The reason even the better-prepared students in the session have typically had severe problems understanding trig identities in high-school is that the "new-from-old" nature of many of these identities has never been pointed out to them. (In fact, it is sometimes clear this this was something not actually understood by the high-school teachers who first introduced them to some of these identities, with students reporting having been asked to enter input into what is actually the "new" side of an identity and use the identity to extract the "resulting" value of what is actually the "old" side of the identity). This aspect of identities doesn't really come up here on page A4, but is worth mentioning as it forms part of the background the students will typically bring to the current section from earlier discussions they will have had. This distinction is particularly important for calculus students since all of the basic "derivative rules" are in fact "new-from-old" identities showing how to get the derivative of a "new" function made up by any of the general complicating steps from simpler, "old" functions, using the "old" information represented by the (assumed to be pre-existing) knowledge about the forms of the derivatives and the values of those "old" functions. Having this perspective, in our experience, dramatically improves the ability of the high fraction of students who did not previously have it to handle effectively the material in the differentiation sections of their first-year calculus courses.